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Frequency formula for a class of fractal vibration system

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ABSTRACT

Four fractal nonlinear oscillators (The fractal Duffing oscillator, fractal attachment oscillator, fractal Toda oscillator, and a fractal nonlinear oscillator) are successfully established by He's fractal derivative in a fractal space, and their variational principles are obtained by semi-inverse transform method. The approximate frequency of the four fractal oscillators are found by a simple frequency formula. The results show the frequency formula is a powerful and simple tool to a class of fractal oscillators.

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1. Introduction

The partial differential equations (PDEs) arise in many fields like the condense matter physics, fluid mechanics, economics and management, etc. There are many methods for solving nonlinear PDEs, for example, the homotopy perturbation method (Anjum & He, 2020a, 2020b; He, 2003; He & El-Dib, 2020; Yu, et al., 2019), variational iteration method (Anjum & He, 2019; He, 1999), Taylor series method (He, 2019,2020a; He & Ji, 2019a; He, et al., 2020), Exp-function method (He, 2013; He & Wu, 2006), and variational-based methods (He, 2020b,2021; He & Ai, 2020). In this paper, we maily study on a class of vibration equation (He, et al., 2021a)

$$mw''+h(w)=0 \tag{1}$$

with the initial condition

$$w(0)=a,\;\; w'(0)=b \tag{2}$$

where m is the mass, h is the nonlinear restoring force, and a and b are constants.

Vibration is the intrinsic property of a packing system, and so far there is no way to stop the vibration, the frequency-amplitude is the main factor for designing a packing system (Song, 2020). The frequency formulation for oscillator (1) was proposed as (He, et al., 2021a)

$$\omega = \sqrt{\frac{h(w)}{w}}\bigg|_{w=NA} \tag{3}$$

where A is the amplitude, it can be approximated calculated as

$$A = \sqrt{a^2 + \frac{b^2}{\omega^2}} \tag{4}$$

and N is $\sqrt{3}/2$ for non-singular oscillators and 0.8 for singular oscillators.

2. Fractal Duffing oscillator

Consider the following well-known Duffing oscillator (He, et al., 2021b)

$$\frac{d^2w}{dt^2} + w + \varepsilon w^3 = 0 \tag{5}$$

with the initial condition

$$w(0) = A, \ \ w'(0) = 0 \tag{6}$$

Variational principle of fractal Duffing oscillator

In a fractal space, Eq. (5) can be described by He's fractal derivative as follows

$$\frac{{}^H D}{Dt^\alpha}\left(\frac{{}^H Dw}{Dt^\alpha}\right) + w + \varepsilon w^3 = 0 \tag{7}$$

with the initial condition

$$w(0) = A, \ \frac{{}^H Dw(0)}{Dt^\alpha} = 0 \tag{8}$$

where ${}^H Dw / Dt^\alpha$ is He's fractal derivative and is defined as follows (He,2018)

$$\frac{{}^H Dw}{Dt^\alpha}(t_0) = \Gamma(1+\alpha) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{w(t) - w(t_0)}{(t-t_0)^\alpha} \tag{9}$$

The variational principle (He, 2020c) of Eq. (7) can be given by semi-inverse transform method as follows

$$J(w) = \int \left\{ \frac{1}{2}\left(\frac{{}^H Dw}{Dt^\alpha}\right)^2 - \frac{1}{2}w^2 - \frac{1}{4}\varepsilon w^4 \right\} {}^H Dt^\alpha \tag{10}$$

Fractal frequency formula

Using the two-scale transform method (He & Ji, 2019b) to Eq. (7) and assume

$$T = t^\alpha \tag{11}$$

therefore, Eq. (7) can be written as follows

$$\frac{{}^H D}{DT}\left(\frac{{}^H Dw}{DT}\right) + w + \varepsilon w^3 = 0 \tag{12}$$

with the initial condition

$$w(0)=A,\ \frac{^{H}Dw(0)}{DT}=0 \tag{13}$$

Adopt the frequency formula (3), and the approximate frequency can be easily obtained as follows

$$\omega=\sqrt{\frac{w+\varepsilon w^3}{w}}\Bigg|_{w=\frac{\sqrt{3}}{2}A}=\sqrt{1+\frac{3}{4}\varepsilon A^2} \tag{14}$$

Table 1. Comparison of Eq. (14) with Eq. (24) in (He, et al., 2021b)

εA^2	0	0.001	0.0025	0.003	0.005	0.007	0.009
Eq.(14)	1	1.000375	1.0009371	1.001124	1.0018724	1.002622	1.003369
Eq.(24) in (He,et al. 2021b)	1	1.000375	1.0009375	1.001125	1.001875	1.002625	1.003375
Exact frequency	1	1.000380	1.0009442	1.00113	1.0018726	1.002613	1.003369

Remark 1.

The Eq. (14) gives more accurate results than Eq.(24) in (He, et al., 2021b) for the fractal Duffing oscillator.

3. Fractal attachment oscillator

Consider the following attachment oscillator (Ren, et al., 2019)

$$\frac{d^2u}{dt^2}+\frac{\varepsilon}{u^3}=0 \tag{15}$$

with the initial condition

$$u(0)=A,\ u'(0)=0 \tag{16}$$

Variational principle of fractal attachment oscillator

In a fractal space, Eq. (15) can be described by He's fractal derivative as follows

$$\frac{^{H}D}{Dt^{\alpha}}\left(\frac{^{H}Du}{Dt^{\alpha}}\right)+\frac{\varepsilon}{u^3}=0 \tag{17}$$

with the initial condition

$$u(0)=A,\ \frac{^{H}Du(0)}{Dt^{\alpha}}=0 \tag{18}$$

where $^{H}Du/Dt^{\alpha}$ is He's fractal derivative and is defined as follows

$$\frac{^{H}Du}{Dt^{\alpha}}(t_0)=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(t)-u(t_0)}{(t-t_0)^{\alpha}} \tag{19}$$

The variational principle of Eq. (17) can be given by semi-inverse transform method as follows

$$J(u)=\int\left\{\frac{1}{2}\left(\frac{^{H}Du}{Dt^{\alpha}}\right)^2+\frac{\varepsilon}{2u^2}\right\}{}^{H}Dt^{\alpha} \tag{20}$$

Fractal frequency formula

Using the two-scale transform method to Eq. (17) and assume

$$T = t^{\alpha} \tag{21}$$

therefore, Eq. (17) can be written as follows

$$\frac{^{H}D}{DT}\left(\frac{^{H}Du}{DT}\right)+\frac{\varepsilon}{u^{3}}=0 \tag{22}$$

with the initial condition

$$u(0)=A,\ \frac{^{H}Du(0)}{DT}=0 \tag{23}$$

Adopt the fractal frequency formula (3), and the approximate frequency can be easily obtained as follows

$$\omega=\sqrt{\frac{\varepsilon}{u^{4}}}\bigg|_{u=0.8A}=1.5625\sqrt{\frac{\varepsilon}{A^{4}}} \tag{24}$$

Remark 2.

In (Ren,et al., 2019), the explicit form of the frequency formula of attachment oscillator is not given.

4. Fractal Toda oscillator

Consider the following Toda oscillator (He, et al., 2021c)

$$\frac{d^{2}u}{dt^{2}}+e^{u}-1=0 \tag{25}$$

with the initial condition

$$u(0)=A,\ u'(0)=0 \tag{26}$$

Variational principle of fractal Toda oscillator

In a fractal space, Eq. (25) can be described by He's fractal derivative as follows

$$\frac{^{H}D}{Dt^{\alpha}}\left(\frac{^{H}Du}{Dt^{\alpha}}\right)+e^{u}-1=0 \tag{27}$$

with the initial condition

$$u(0)=A,\ \frac{^{H}Du(0)}{Dt^{\alpha}}=0 \tag{28}$$

where $^{H}Du/Dt^{\alpha}$ He's fractal derivative and is defined as follows

$$\frac{^{H}Du}{Dt^{\alpha}}(t_0)=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(t)-u(t_0)}{(t-t_0)^{\alpha}} \tag{29}$$

The variational principle of Eq. (27) can be given by semi-inverse transform method as follows

$$J(u)=\int\left\{\frac{1}{2}\left(\frac{^{H}Du}{Dt^{\alpha}}\right)^{2}+e^{u}-u\right\}{}^{H}Dt^{\alpha} \tag{30}$$

Fractal frequency formula

Using the two-scale transform method to Eq. (27) and assume

$$T = t^{\alpha} \tag{31}$$

therefore, Eq. (27) can be written as follows

$$\frac{{}^{H}D}{DT}\left(\frac{{}^{H}Du}{DT}\right) + e^{u} - 1 = 0 \tag{32}$$

with the initial condition

$$u(0) = A,\ \frac{{}^{H}Du(0)}{DT} = 0 \tag{33}$$

Adopt the frequency formula (3), and the approximate frequency can be easily obtained as follows

$$\omega = \sqrt{\frac{e^{u}-1}{u}}\Bigg|_{u=\frac{\sqrt{3}}{2}A} = \frac{2(-1+e^{\frac{\sqrt{3}A}{2}})}{\sqrt{3}A} \tag{34}$$

Remark 3.

In (He, et al., 2021c), the frequency formula (34) is not given for Toda oscillator.

5. A fractal nonlinear oscillator

Consider the following nonlinear oscillator (He, 2014)

$$\frac{d^2u}{dt^2} + u + \varepsilon_1 u^2 + \varepsilon_2 u^3 = 0 \tag{35}$$

with the initial condition

$$u(0) = A,\ u'(0) = 0 \tag{36}$$

Variational principle of a fractal nonlinear oscillator

In a fractal space, Eq. (35) can be described by He's fractal derivative as follows

$$\frac{{}^{H}D}{Dt^{\alpha}}\left(\frac{{}^{H}Du}{Dt^{\alpha}}\right) + u + \varepsilon_1 u^2 + \varepsilon_2 u^3 = 0 \tag{37}$$

with the initial condition

$$u(0) = A,\ \frac{{}^{H}Du(0)}{Dt^{\alpha}} = 0 \tag{38}$$

where ${}^{H}Du / Dt^{\alpha}$ He's fractal derivative and is defined as follows

$$\frac{{}^{H}Du}{Dt^{\alpha}}(t_0) = \Gamma(1+\alpha) \lim_{\substack{t-t_0\to\Delta t \\ \Delta t\neq 0}} \frac{u(t)-u(t_0)}{(t-t_0)^{\alpha}} \tag{39}$$

The variational principle of Eq. (37) can be given by semi-inverse transform method as follows

$$J(u) = \int \left\{ \frac{1}{2}\left(\frac{{}^{H}Du}{Dt^{\alpha}}\right)^2 - \frac{1}{2}u^2 + \frac{1}{3}\varepsilon_1 u^3 - \frac{1}{4}\varepsilon_2 u^4 \right\} {}^{H}Dt^{\alpha} \tag{40}$$

Fractal frequency formula

Using the two-scale transform method to Eq. (37) and assume

$$T = t^{\alpha} \tag{41}$$

therefore, Eq. (37) can be written as follows

$$\frac{{}^{H}D}{DT}\left(\frac{{}^{H}Du}{DT}\right) + u + \varepsilon_1 u^2 + \varepsilon_2 u^3 = 0 \tag{42}$$

with the initial condition

$$u(0) = A,\ \frac{{}^{H}Du(0)}{DT} = 0 \tag{43}$$

Adopt the frequency formula (3), and the approximate frequency can be easily obtained as follows

$$\omega = \sqrt{1 + \varepsilon_1 u + \varepsilon_2 u^2}\Big|_{u=\frac{\sqrt{3}}{2}A} = \sqrt{1 + \frac{\sqrt{3}}{2}A\varepsilon_1 + \frac{3}{4}A^2\varepsilon_2} \tag{44}$$

We write down Nayfeh's result (He, 2014) for comparison

$$\omega = 1 + \left(\frac{3}{8}\varepsilon_2 - \frac{5}{12}{\varepsilon_1}^2\right)A^2 \tag{45}$$

Table 2. Comparison of Eq. (44) with Eq. (45)

A	ε_1	ε_2	Eq.(44)	Eq.(45)	Relative error
1	0.001	0.001	1.00081	1.00037	0.044%
1	0.01	0.05	1.02282	1.01871	0.4%
10	0.0001	0.0005	1.019	1.01875	0.025%
10	0.001	0.0025	1.09369	1.09371	0.0018%
100	0.01	0.0005	2.36982	2.45833	3.7%
100	0.005	0.00025	1.81879	1.83333	0.8%

Remark 4.

Table 2 shows the good agreement between Eq. (44) and Eq. (45).

6. Conclusions

In this paper, four nonlinear oscillators are described by He's fractal derivative in a fractal space, and their variational principle are successfully established via semi-inverse transform method. The two-scale transform method and fractal frequency formulas are adopted to find the approximate frequency of fractal oscillator equation. The examples show the frequency formula is simple and effective.

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